THE RICCI FLOW IN A CLASS OF SOLVMANIFOLDS.

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ABSTRACT. In this paper, we study the Ricci flow of solvmanifolds whose Lie algebra has an abelian ideal of codimension one, by using the bracket flow. We prove that solutions to the Ricci flow are immortal, the ω-limit of bracket flow solutions is a single point, and that for any sequence of times there exists a subsequence in which the Ricci flow converges, in the pointed topology, to a manifold which is locally isometric to a flat manifold. We give a functional which is non-increasing along a normalized bracket flow that will allow us to prove that given a sequence of times, one can extract a subsequence converging to an algebraic soliton, and to determine which of these limits are flat. Finally, we use these results to prove that if a Lie group in this class admits a Riemannian metric of negative sectional curvature, then the curvature of any Ricci flow solution will become negative in finite time.

1. INTRODUCTION

The Ricci flow is an evolution equation for a curve of Riemannian metrics on a manifold. In recent years, the Ricci flow has proven to be a very important tool. Many strong results, not only in Riemannian geometry, have been proven by using this equation. The objective of this paper is to study the Ricci flow for solvmanifolds whose Lie algebra has an abelian ideal of codimension one and get similar results to those obtained by J. Lauret in [L2] in the case of nilmanifolds.

Let (G, g) be a solvmanifold, i.e. a simply connected solvable Lie group G endowed with a left-invariant metric g. Assume that the Lie algebra of G has an abelian ideal of codimension one. Consider the Ricci flow starting at g, that is,

$$\tfrac{\partial}{\partial t} g(t) = -2 \operatorname{Rc}(g(t)), \quad g(0) = g.$$

The solution $g(t)$ is a left-invariant metric for all t, thus each $g(t)$ is determined by an inner product on the Lie algebra. We will follow the approach in [L4] to study the evolution of these metrics by varying Lie brackets instead of inner products.

More precisely, let μ be a Lie bracket on $\mathbb{R}^{n+1}$ with an abelian ideal of codimension one. We may assume that μ is determined by $A = \operatorname{ad}_{\mu}(e_0)|_{\mathbb{R}^n} \in \mathfrak{gl}_n(\mathbb{R})$, where $\mathbb{R}^{n+1} = \mathbb{R} e_0 \oplus \mathbb{R}^n$ and $\mathbb{R}^n$ is the abelian ideal, and so it will be denoted by μ_A. Each μ_A determines a Riemannian manifold (G_{μ_A}, g_{μ_A}), where G_{μ_A} is the simply connected Lie group with Lie algebra $(\mathbb{R}^{n+1}, \mu_A)$ and g_{μ_A} is the left-invariant metric determined by $\langle \cdot, \cdot \rangle$, the canonical inner product on $\mathbb{R}^{n+1}$. Every solvmanifold whose Lie algebra has an abelian ideal of codimension one is isometric to some μ_A (see Section 2). By [L4, Theorem 3.3], the Ricci flow solution is given by $g(t) = \varphi(t)^* g_{\mu(t)}$, where $\mu(t)$ is a family of Lie brackets solving a ODE called the bracket flow, and $\varphi(t) : G \to G_{\mu(t)}$ is the Lie group isomorphism with derivative $h(t) : (\mathbb{R}^{n+1}, \mu) \to (\mathbb{R}^{n+1}, \mu(t))$, and $h(t)$ satisfies

$$\tfrac{d}{dt} h = -h \operatorname{Ric}(\langle \cdot, \cdot \rangle_t), \quad \tfrac{d}{dt} h = -\operatorname{Ric}_{\mu(t)} h, \quad h(0) = I.$$

In our case, we see that $\mu(t) = \mu_{A(t)}$, where $A(t) \in \mathfrak{gl}_n(\mathbb{R})$ is the solution to the following ODE,

$$\tfrac{d}{dt} A = -\operatorname{tr}(S(A)^2) A + \tfrac{1}{2}[A, [A, A^t]] - \tfrac{1}{2} \operatorname{tr}(A)[A, A^t], \quad A(0) = A,$$

and then we study the evolution of the matrix A. The main results in this paper can be summarized as follows:

- The Ricci flow solution $g(t)$ is defined for all $t \in (T_-, \infty)$, where $-\infty < T_- < 0$, and if $\operatorname{tr}(A^2) \geq 0$, then $g(t)$ is a Type-III solution (see Proposition 3.4 and Proposition 3.14).
- The scaling-invariant functional $\frac{\|[A(t),A(t)^t]\|}{\|A(t)\|^2}$ is strictly decreasing unless μ_A is an algebraic soliton, in which case it is constant (see Lemma 3.6). This happens precisely when A is either normal or nilpotent of a special kind (see Proposition 3.3).
- For any sequence $t_k \to \infty$, there exists a subsequence of $(G_{\mu_{A(t_k)}}, g_{\mu_{A(t_k)}})$ which converges in the pointed topology to a flat manifold, up to local isometry (see Corollary 3.11).
- If $\operatorname{tr}(A) = 0$ (i.e. G_{μ_A} unimodular), then $B(t) = \frac{A(t)}{\|A(t)\|}$ converges to a matrix B_∞, as $t \to \infty$ (see Lemma 4.1 and Remark 4.2).
- For any sequence $t_k \to \infty$, there exists a subsequence of $(G_{\mu_{B(t_k)}}, g_{\mu_{B(t_k)}})$ which converges in the pointed topology to $(G_{\mu_{B_\infty}}, g_{\mu_{B_\infty}})$ (up to local isometry), which is an algebraic soliton. In addition, $(G_{\mu_{B_\infty}}, g_{\mu_{B_\infty}})$ is non-flat, unless every eigenvalue of A is purely imaginary (see Theorem 5.2).
- If G_{μ_A} admits a negatively curved left-invariant metric, then there exists $t_0 > 0$ such that $g(t)$ is negatively curved for all $t \geq t_0$ (see Theorem 6.5). This is not true in general for solvmanifolds (see Example 6.6).

2. Preliminaries

2.1. **The Ricci flow.** Let (M, g) be a Riemannian manifold. The Ricci flow starting at (M, g) is the following partial differential equation:

$$\text{(1)} \qquad \frac{\partial}{\partial t} g(t) = -2 \operatorname{Rc}(g(t)), \quad g(0) = g,$$

where $g(t)$ is a curve of Riemannian metrics on M and $\operatorname{Rc}(g(t))$ the Ricci tensor of the metric $g(t)$.

A complete Riemannian metric g on a differentiable manifold M is a Ricci soliton if its Ricci tensor satisfies

$$\operatorname{Rc}(g) = cg + L_X g, \text{ for some } c \in \mathbb{R}, \ X \in \chi(M) \text{ complete},$$

where $\chi(M)$ denotes the space of differentiable vector fields on M and L_X the usual Lie derivative in the direction of the field X.

Equivalently, Ricci solitons are precisely the metrics that evolve along the Ricci flow only by the action of diffeomorphisms and scaling (i.e. $g(t) = c(t)\varphi(t)^* g$), giving geometries that are equivalent to the starting point, for all time t (see [C] for more information about Ricci solitons).

Definition 2.1. A Ricci flow solution $g(t)$ is said to be of Type-III if it is defined for $t \in [0, \infty)$ and there exists $C \in \mathbb{R}$ such that

$$\|\operatorname{Rm}(g(t))\| \leq \tfrac{C}{t}, \qquad \forall t \in (0, \infty),$$

where $\operatorname{Rm}(g(t))$ is the Riemann curvature tensor of the metric $g(t)$.

2.2. **Varying Lie brackets.** We fix $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$, with $\langle \cdot, \cdot \rangle$ an inner product on $\mathbb{R}^n$ and we define

$$\mathfrak{L}_n = \{\mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n : \mu \text{ is bilinear, skew-symmetric and satisfies Jacobi}\},$$

$$\mathcal{N}_n = \{\mu \in \mathfrak{L}_n : \mu \text{ is nilpotent}\},$$

and $\operatorname{ad}_\mu : \mathbb{R}^n \to \mathbb{R}^n$ the adjoint representation of $\mu \in \mathfrak{L}_n$ (i.e. $\operatorname{ad}_\mu(x)(y) = \mu(x, y)$).

Then, $GL_n(\mathbb{R})$ acts on $\mathfrak{L}_n$ by

$$(2) \qquad h.\mu(X,Y) = h\mu(h^{-1}X, h^{-1}Y), \quad X,Y \in \mathbb{R}^n, \quad h \in GL_n(\mathbb{R}), \quad \mu \in \mathfrak{L}_n.$$

Each $\mu \in \mathfrak{L}_n$ defines a Lie group endowed with a left-invariant Riemannian metric,

$$\mu \in \mathfrak{L}_n \rightsquigarrow (G_\mu, \langle \cdot, \cdot \rangle),$$

where G_μ is the simply connected Lie group with Lie algebra $(\mathbb{R}^n, \mu)$ endowed with the left-invariant Riemannian metric determined by the inner product $\langle \cdot, \cdot \rangle$. Often, we will denote this metric by g_μ. Note that g_μ may be viewed as a metric on $\mathbb{R}^n$, in fact, G_μ is diffeomorphic to $\mathbb{R}^n$.

Geometrically, each $h \in GL_n(\mathbb{R})$ determines a Riemannian isometry

$$(3) \qquad (G_{h.\mu}, \langle \cdot, \cdot \rangle) \to (G_\mu, \langle h\cdot, h\cdot \rangle),$$

by exponentiating the Lie algebra isomorphism $h^{-1} : (\mathbb{R}^n, h.\mu) \to (\mathbb{R}^n, \mu)$. Thus the orbit $GL_n(\mathbb{R}).\mu$ parameterizes the set of all left-invariant metrics on G_μ.

Definition 2.2. Let (G, g) be a Lie group with a left-invariant Riemannian metric; g is called an algebraic soliton if

$$(4) \qquad \operatorname{Ric}(g) = cI + D, \quad \text{for some} \quad c \in \mathbb{R}, \quad D \in \operatorname{Der}(\mathfrak{g}),$$

where $\operatorname{Ric}(g)$ is the Ricci operator of g and $\mathfrak{g}$ is the Lie algebra of G.

Any homogeneous simply connected algebraic soliton is a Ricci soliton (see [LL, Proposition 3.3]).

2.3. **Ricci flow on Lie groups and the bracket flow.** Let (G, g) be a simply connected Lie group endowed with a left-invariant Riemannian metric. Then, if we fix $\langle \cdot, \cdot \rangle$ an inner product on the Lie algebra of G, (G, g) is isometric to (G_μ, g_μ), for some $\mu \in \mathcal{L}_n$. In this case, the equation of the Ricci flow (1) is equivalent to the following ordinary differential equation (see [L4, Section 3]):

$$(5) \qquad \frac{d}{dt}\langle \cdot, \cdot \rangle_t = -2\operatorname{Rc}(\langle \cdot, \cdot \rangle_t), \quad \langle \cdot, \cdot \rangle_0 = \langle \cdot, \cdot \rangle,$$

where $\operatorname{Rc}(\langle \cdot, \cdot \rangle_t) := \operatorname{Rc}(g(t))(e)$ and e is the identity of G_μ. In Subsection 2.2, we have observed that $GL_n(\mathbb{R}).\mu$ parameterizes the set of all left invariant Riemannian metrics on G_μ, then it is very natural to ask: How is the behavior of the Ricci flow in $\mathfrak{L}_n$?

Definition 2.3. Given $\mu \in \mathcal{L}_n$, the bracket flow starting at μ is the following ordinary differential equation:

$$(6) \qquad \frac{d}{dt}\mu(t) = \delta_{\mu(t)}\left(\operatorname{Ric}_{\mu(t)}\right), \quad \mu(0) = \mu,$$

where $\delta_\mu(A) = \mu(A\cdot, \cdot) + \mu(\cdot, A\cdot) - A\mu(\cdot, \cdot)$, $A \in GL_n(\mathbb{R})$, $\mu \in \mathcal{L}_n$.

Let us consider $g(t)$ the Ricci flow solution flow starting at g_μ, and $\mu(t)$ the solution of the bracket flow starting at μ. By [L4], we know that $g(t)$ and $\mu(t)$ are related in the following way.

Theorem 2.4. [L4, Theorem 3.3] *There exists time-dependent diffeomorphisms*

$$\varphi(t) : G \to G_{\mu(t)} \text{ such that } g(t) = \varphi(t)^* g_{\mu(t)}, \quad \forall t \in (a,b).$$

Moreover, if we identify $G = G_\mu$, then $\varphi(t) : G_\mu \to G_{\mu(t)}$ can be chosen as the equivariant diffeomorphism determined by the Lie group isomorphism between G_μ and $G_{\mu(t)}$ with derivative $h : \mathbb{R}^n \to \mathbb{R}^n$, where $h(t) := d\varphi(t)\mid_e : \mathbb{R}^n \to \mathbb{R}^n$ is the solution to any of the following systems of ordinary differential equations:

(1) $\frac{d}{dt}h = -h\operatorname{Rc}(\langle \cdot, \cdot \rangle_t)$, $h(0) = I$.

(2) $\frac{d}{dt}h = -\operatorname{Ric}_{\mu(t)} h$, $h(0) = I$.

The following conditions hold:

(3) $\langle \cdot, \cdot \rangle_t = \langle h\cdot, h\cdot \rangle$.
(4) $\mu(t) = h\mu_0(h^{-1}\cdot, h^{-1}\cdot)$.

Remark 2.5. In this paper, Theorem 2.4 has only been stated in the case of Lie groups, however, in [L4] it is stated and proved in the general homogeneous case.

So, the Ricci flow $g(t)$ can be obtained from the bracket flow $\mu(t)$ by solving (2) and applying part (3). In the same way, we can obtain $\mu(t)$ solving (1) and replacing in (4). In particular, both flows are defined in the same interval of time. For more information, see [L4].

We now recall some results proved by J. Lauret in [L2] about the Ricci flow for simply connected nilmanifolds.

Theorem 2.6. [L2] *Let $\mu(t)$ be the solution bracket flow starting at $\mu \in \mathcal{N}_n$, and $g(t)$ the Ricci flow starting at g_μ. Then*

(i) $\mu(t)$ *is defined for all $t \in [0, \infty)$.*
(ii) $g(t)$ *is a Type-III solution for a constant C_n that only depends on the dimension n.*
(iii) $\mu(t) \to 0$, *as $t \to \infty$. Moreover, $g_{\mu(t)}$ converges in $\mathcal{C}^\infty$ to the flat metric g_0.*
(iv) $g_{\frac{\mu(t)}{\|\mu(t)\|}}$ *converges in $\mathcal{C}^\infty$ to an algebraic soliton g_λ uniformly on compact sets in $\mathbb{R}^n$, as $t \to \infty$.*

3. The bracket flow in a class of solvmanifolds

In this section, we study the bracket flow for a metric solvable Lie algebra with an abelian ideal of codimension one.

We consider $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)$, with $\langle \cdot, \cdot \rangle$ the canonical inner product on $\mathbb{R}^{n+1}$. If the dimension of the Lie algebra is $n + 1$, then up to isomorphism, we can assume that the Lie bracket has the following form with respect to the canonical basis $\{e_0, e_1, \dots, e_n\}$:

$$\mu_A(e_0, e_i) = Ae_i, \ \ i = 1, \dots, n, \quad \mu_A(e_i, e_j) = 0, \ \ \forall i, j \geq 1, \quad A \in \mathfrak{gl}_n(\mathbb{R}),$$

where we think of an $A \in \mathfrak{gl}_n(\mathbb{R})$ as an operator acting on $\mathbb{R}^n$, the subspace generated by $\{e_1, e_2, \dots, e_n\}$ (i.e. the codimension-one abelian ideal). From now on, we denote these algebras by $(\mathbb{R}^{n+1}, \mu_A)$, or simply, μ_A.

Lemma 3.1. *If $A_0 \in \mathfrak{gl}_n(\mathbb{R})$, then the bracket flow starting at μ_{A_0} is given by $\mu_{A(t)}$, $t \in (T_-, T_+)$, where $A(t)$ satisfies*

$$(7) \qquad \frac{d}{dt}A = -\operatorname{tr}(S(A)^2)A + \tfrac{1}{2}[A, [A, A^t]] - \tfrac{1}{2}\operatorname{tr}(A)[A, A^t], \quad A(0) = A_0.$$

Proof. By using the formula for the Ricci operator of a solvmanifold (see for instance [L1, Section 4]), we obtain that the Ricci operator of (G_{μ_A}, g_{μ_A}) with respect to the basis $\{e_0, e_1, \dots, e_n\}$ is represented by the matrix

$$(8) \qquad \operatorname{Ric}_{\mu_A} = \begin{pmatrix} -\operatorname{tr}(S(A)^2) & 0 \\ 0 & \tfrac{1}{2}[A, A^t] - \operatorname{tr}(A)S(A) \end{pmatrix},$$

where $S(A) = \tfrac{1}{2}(A + A^t)$ is the symmetric part of the matrix A and $\operatorname{tr}(A)$ is the trace. Then,

$$
\begin{aligned}
\delta_{\mu_A}(\operatorname{Ric}_{\mu_A})(e_0, e_i) &= \mu_A(\operatorname{Ric}_{\mu_A}e_0, e_i) + \mu_A(e_0, \operatorname{Ric}_{\mu_A}e_i) - \operatorname{Ric}_{\mu_A}\mu_A(e_0, e_i) \\
&= -\operatorname{tr}(S(A)^2)Ae_i + A\operatorname{Ric}_{\mu_A}|_{\mathbb{R}^n}e_i - \operatorname{Ric}_{\mu_A}|_{\mathbb{R}^n}Ae_i \\
&= -\operatorname{tr}(S(A)^2)Ae_i + [A, \operatorname{Ric}_{\mu_A}|_{\mathbb{R}^n}]e_i \\
&= \left(-\operatorname{tr}(S(A)^2)A + \tfrac{1}{2}[A, [A, A^t]] - \tfrac{1}{2}\operatorname{tr}(A)[A, A^t]\right)e_i,
\end{aligned}
$$

and, on the other hand, we have that $\delta_{\mu_A}(\operatorname{Ric}_{\mu_A})(e_i, e_j) = 0$, for all $i, j \geq 1$, as $\mu_A|_{\mathbb{R}^n \times \mathbb{R}^n} = 0$. So,

$$\delta_{\mu_A}(\operatorname{Ric}_{\mu_A}) = \mu_B, \quad \text{where} \quad B = -\operatorname{tr}(S(A)^2)A + \tfrac{1}{2}[A, [A, A^t]] - \tfrac{1}{2}\operatorname{tr}(A)[A, A^t].$$

Then, this family of Lie algebras is invariant under the bracket flow, which is equivalent to (7). In addition, the maximal interval of time where $\mu_{A(t)}$ exists is of the form (T_-, T_+) for some $-\infty \leq T_- < 0 < T_+ \leq \infty$, since (7) is an ODE. $\qquad\square$

So, given a matrix A_0, we have that the bracket flow starting at μ_{A_0} is equivalent to an evolution equation for a curve of matrices with initial condition A_0. In what follows, we will often think of the bracket flow as this evolution.

Remark 3.2. Note that the only fixed points of the system (7) are the skew-symmetric matrices, which are precisely the flat solvmanifolds of the form μ_A, since by (8) they are precisely the Ricci-flat ones (see [AK] and [Mil]).

Proposition 3.3. *For any $A_0 \in \mathfrak{gl}_n(\mathbb{R})$, the following conditions are equivalent:*
  (i) *μ_{A_0} is an algebraic soliton.*
  (ii) *A_0 is either a normal matrix or A_0 is a nilpotent matrix such that $[A_0, [A_0, A_0{}^t]] = cA_0$, for some $c \in \mathbb{R}$.*

Moreover, the evolution of the bracket flow is respectively given by

$$A(t) = \left(2\operatorname{tr}(S(A_0)^2)t + 1\right)^{-1/2} A_0 \quad or \quad A(t) = \left((-\|A_0\|^2 + c)t + 1\right)^{-1/2} A_0.$$

Proof. Assuming part (i), we have two cases:
  - If the nilradical of μ_{A_0} has dimension n, then A_0 is a normal matrix (see [L1, Theorem 4.8]).
  - If the nilradical of μ_{A_0} has dimension $n+1$, then μ_{A_0} is nilpotent and so A_0 is a nilpotent matrix. In addition, from (8), we have that

$$\begin{pmatrix} -\frac{1}{2}\|A_0\|^2 & 0 \\ 0 & \frac{1}{2}[A_0, A_0{}^t] \end{pmatrix} = \operatorname{Ric}_{\mu_{A_0}} = cI + D,$$

and it follows that $D(e_0) = \lambda e_0$. Also, we know that $[\operatorname{ad}_{\mu_{A_0}}(e_0), D] = -\operatorname{ad}_{\mu_{A_0}}(D(e_0))$, so

$$\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}[A_0, [A_0, A_0{}^t]] \end{pmatrix} = [\operatorname{ad}_{\mu_{A_0}}(e_0), \operatorname{Ric}_{\mu_{A_0}}] = [\operatorname{ad}_{\mu_{A_0}}(e_0), D] = -\lambda \operatorname{ad}_{\mu_{A_0}}(e_0) = -\lambda \begin{pmatrix} 0 & 0 \\ 0 & A_0 \end{pmatrix}.$$

Conversely, if A_0 is a normal matrix, then μ_{A_0} is an algebraic soliton (see [L1, Theorem 4.8]) and if A_0 is a nilpotent matrix which satisfies $[A_0, [A_0, A_0{}^t]] = cA_0$, then

$$\operatorname{Ric}_{\mu_{A_0}} = \begin{pmatrix} -\frac{1}{2}\|A_0\|^2 & 0 \\ 0 & \frac{1}{2}[A_0, A_0{}^t] \end{pmatrix} = \frac{c - \|A_0\|^2}{2}I + \begin{pmatrix} -\frac{c}{2} & 0 \\ 0 & -\frac{c}{2}I + \frac{1}{2}\|A_0\|^2 I + \frac{1}{2}[A_0, A_0{}^t] \end{pmatrix},$$

and it is easy to see that $\begin{pmatrix} -\frac{c}{2} & 0 \\ 0 & -\frac{c}{2}I + \frac{1}{2}\|A_0\|^2 I + \frac{1}{2}[A_0, A_0{}^t] \end{pmatrix}$ is a derivation of μ_{A_0}, and so (i) is proved.

Finally, if μ_{A_0} is an algebraic soliton, then the family $A(t) = a(t)A_0$ is invariant under the flow. Therefore, we have that
  - If A_0 is a normal matrix, then the bracket flow is equivalent to the following differential equation for $a = a(t)$

$$a' = -\operatorname{tr}(S(A_0)^2)a^3, \quad a(0) = 1,$$

  and so the solution is $A(t) = (2\operatorname{tr}(S(A_0)^2)t + 1)^{-1/2}A_0$.
  - If A_0 is a nilpotent matrix, then the bracket flow is equivalent to

$$a' = \frac{-\|A_0\|^2 + c}{2}a^3, \quad a(0) = 1,$$

  and so the solution is $A(t) = ((-\|A_0\|^2 + c)t + 1)^{-1/2}A_0$.

$\qquad\square$

The first natural question that arises is related with the maximal time interval of the solution $A(t)$. An important point to observe here is that $-\infty < T_-$ since $(G_{\mu_{A(t)}}, g_{\mu_{A(t)}})$ always has non-positive scalar curvature (see (8)).

Proposition 3.4. $A(t)$ *is always defined for all* $t \in [0, \infty)$ *(i.e.* $T_+ = \infty$*).*

Proof. By using (7), we get

$$\tfrac{d}{dt}\|A\|^2 = 2\langle A, -\operatorname{tr}(S(A)^2)A\rangle + 2\langle A, \tfrac{1}{2}[A, [A, A^t]]\rangle - 2\langle A, \tfrac{1}{2}\operatorname{tr}(A)[A, A^t]\rangle.$$

But since $\langle A, [A, [A, A^t]]\rangle = -\|[A, A^t]\|^2$ and $\langle A, [A, A^t]\rangle = 0$, it follows that

$$(9) \qquad\qquad \tfrac{d}{dt}\|A\|^2 = -2\operatorname{tr}(S(A)^2)\|A\|^2 - \|[A, A^t]\|^2 \le 0.$$

Therefore, $\|A\|^2$ decreases and so $A(t)$ is defined for all $t \in [0, \infty)$, as the solution remains in a compact subset. $\qquad\square$

Remark 3.5. By Theorem 2.4 and the previous proposition, we obtain that the Ricci flow starting at any of these solvmanifolds $(G_{\mu_{A_0}}, \langle \cdot, \cdot \rangle)$ is defined for $t \in [0, \infty)$, often called an immortal solution.

In what follows, we introduce a positive, non-increasing function along the normalized bracket flow $\frac{A(t)}{\|A(t)\|}$, which is strictly decreasing unless $\mu_{\frac{A_0}{\|A_0\|}}$ is an algebraic soliton. The advantage of having this function lies in the fact that it will allow us to prove that for any sequence $t_k \to \infty$ there exists a subsequence in which the normalized bracket flow always converges to an algebraic soliton.

Lemma 3.6. *Let* $\mu_{A(t)}$ *be the bracket flow starting at* μ_{A_0} *and set* $B(t) = \frac{A(t)}{\|A(t)\|}$. *Then* $F(B) = \|[B, B^t]\|^2$ *is a positive, non-increasing function along the flow. Moreover,* $\frac{d}{dt}|_{t=t_0} F(B) = 0$, *for some* t_0, *if and only if* $\mu_{B(0)}$ *is an algebraic soliton.*

Proof. We consider $F : \mathfrak{gl}_n(\mathbb{R}) \to \mathbb{R}$,

$$F(C) = \|[C, C^t]\|^2.$$

Then

$$\tfrac{d}{dt} F\left(\tfrac{A}{\|A\|}\right) = \tfrac{d}{dt}\tfrac{\|[A, A^t]\|^2}{\|A\|^4} = \tfrac{1}{\|A\|^8}(\|A\|^4 \tfrac{d}{dt}\|[A, A^t]\|^2 - \|[A, A^t]\|^2 \tfrac{d}{dt}\|A\|^4).$$

By using the bilinearity of the inner product and the lie bracket we obtain that

$$\tfrac{d}{dt}\|[A, A^t]\|^2 = -4\operatorname{tr}(S(A)^2)\|[A, A^t]\|^2 - 2\|[A, [A, A^t]]\|^2,$$

and from (9)

$$\tfrac{d}{dt}\|A\|^4 = 2\|A\|^2 \tfrac{d}{dt}\|A\|^2 = -4\operatorname{tr}(S(A)^2)\|A\|^4 - 2\|A\|^2\|[A, A^t]\|^2.$$

Then, if we consider $B = \frac{A}{\|A\|}$, it follows that

$$(10) \qquad\qquad \tfrac{d}{dt} F\left(\tfrac{A}{\|A\|}\right) = 2\|A\|^2(\|[B, B^t]\|^4 - \|B\|^2\|[B, [B, B^t]]\|^2) \le 0,$$

by using the Cauchy-Schwarz inequality. Moreover, if there exists $t_0 \in \mathbb{R}$ such that $\frac{d}{dt}|_{t=t_0} F\left(\frac{A}{\|A\|}\right) = 0$, then the Cauchy-Schwarz equality holds and there exists $c \in \mathbb{R}$ such that

$$(11) \qquad\qquad [B(t_0), [B(t_0), B(t_0)^t]] = cB(t_0).$$

We have two cases:

- If $c = 0$, then $[B(t_0), [B(t_0), B(t_0)^t]] = 0$, and so $\mathrm{tr}([B(t_0), [B(t_0), B(t_0)^t]]B(t_0)^t) = 0$, this implies that $\|[B(t_0), B(t_0)^t]\|^2 = 0$, i.e. $B(t_0)$ is normal and $\mu_{B(t_0)}$ is an algebraic soliton (see Proposition 3.3). On the other hand, by using (7) and (9), it is easy to see that

$$
(12) \qquad
\begin{aligned}
\tfrac{d}{dt} B &= \tfrac{d}{dt} \tfrac{A}{\|A\|} = \tfrac{1}{2\|A\|}\left([A, [A, A^t]] - \mathrm{tr}(A)[A, A^t] + \tfrac{\|[A, A^t]\|^2}{\|A\|^2} A\right) \\
&= \tfrac{\|A\|^2}{2}\left([B, [B, B^t]] - \mathrm{tr}(B)[B, B^t] + \|[B, B^t]\|^2 B\right),
\end{aligned}
$$

so, $B(t) = B(t_0)$, for all t, since $B(t_0)$ is a fixed point of (12). It follows that $\mu_{B(t)} = \mu_{B(t_0)}$ for all t.

- If $c \neq 0$, then by using (11), we obtain that $\mathrm{tr}(B(t_0)) = 0$ and $\mathrm{tr}(B(t_0)^k) = 0$, since

$$
\begin{aligned}
c\,\mathrm{tr}(B(t_0)^{k+1}) &= \mathrm{tr}([B(t_0), [B(t_0), B(t_0)^t]]B(t_0)^k) \\
&= \mathrm{tr}([B(t_0)^k, B(t_0)][B(t_0), B(t_0)^t]) \\
&= 0.
\end{aligned}
$$

Therefore, $B(t_0)$ is a nilpotent matrix that satisfies (11), so by Proposition 3.3 we have that $\mu_{B(t_0)}$ is an algebraic soliton. In addition, $B(t_0)$ is a fixed point of (12), so, $\mu_{B(t)} = \mu_{B(t_0)}$ for all t.

Conversely, if $\mu_{B(0)}$ is an algebraic soliton, then by using (10), we have that $\frac{d}{dt} F\left(\frac{A}{\|A\|}\right) = 0$. $\square$

Corollary 3.7. *Let $\mu_{A(t)}$ be the bracket flow starting at μ_{A_0} and set $B(t) = \frac{A(t)}{\|A(t)\|}$. Then for any sequence $t_k \to \infty$ there exists a subsequence of $(G_{\mu_{B(t_k)}}, g_{\mu_{B(t_k)}})$ converging in the pointed topology to an algebraic soliton $(G_{\mu_{B_\infty}}, g_{\mu_{B_\infty}})$.*

Proof. Every sequence $B(t_k)$ has a convergent subsequence, i.e. after passing to a subsequence, $B(t_k)$ converges to a matrix B_∞. Then μ_{B_∞} is an algebraic soliton by Lemma 3.6, as B_∞ is a fixed point of the flow. $\square$

From now on, our purpose is to study the ODE (7). We emphasize that our aim is not to solve the ODE, we are interested in understanding the qualitative behavior of the solution along the time, which is not trivial to predict even when n is very small. In the next lemma we study how it evolves.

Lemma 3.8. *The bracket flow $\mu_{A(t)}$ starting at μ_{A_0} has the form*

$$
(13) \qquad A(t) = a(t)\varphi_t A_0 \varphi_t^{-1},
$$

where $a(t)$ is a positive, non-increasing, real valued function, and $\varphi_t \in \mathrm{GL}_n(\mathbb{R})$ for each t.

Proof. If $h(t) = \begin{pmatrix} b(t) & 0 \\ 0 & \varphi_t \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbb{R})$, with $b(t)$ a real function and $\varphi_t \in \mathrm{GL}_n(\mathbb{R})$, then

$$
-\mathrm{Ric}_{\mu_{A(t)}} h(t) = -\begin{pmatrix} -\mathrm{tr}(S(A(t))^2)b(t) & 0 \\ 0 & (\tfrac{1}{2}[A(t), A(t)^t] - \mathrm{tr}(A(t))S(A(t)))\varphi_t \end{pmatrix}.
$$

The map h given in part (2) of Theorem 2.4 has therefore the form

$$
h(t) = \begin{pmatrix} b(t) & 0 \\ 0 & \varphi_t \end{pmatrix},
$$

and it follows from (4) of the same theorem that

$$
\mu_{A(t)} = h(t).\mu_{A_0} = \mu_{\frac{1}{b(t)}\varphi_t A_0 \varphi_t^{-1}}.
$$

In addition,

$$
\begin{cases} b'(t) = \mathrm{tr}(S(A(t))^2)b(t), \\ b(0) = 1, \end{cases}
$$

so, we have that b is a positive, non-decreasing function. It follows that if $a(t) = \frac{1}{b(t)}$, then $a(t)$ is a positive, non-increasing function. $\qquad\square$

In what follows, $\mu_{A(t)}$, $A(t) \in \mathfrak{gl}_n(\mathbb{R})$, will be the bracket flow solution starting at μ_{A_0} and we will denote it simply by $A(t)$.

Proposition 3.9. *Assume that $A(t_k) \to A_\infty$, for some sequence $t_k \to \infty$. Then $\mathrm{Spec}(A_\infty) = a_\infty \mathrm{Spec}(A_0)$, for some $a_\infty \in \mathbb{R}$. Here $\mathrm{Spec}(B)$ denotes the unordered set of n complex eigenvalues of the matrix $B \in \mathfrak{gl}_n(\mathbb{R})$.*

Proof. We know that $A(t) = a(t)\varphi_t A_0 \varphi_t^{-1}$ by Lemma 3.8, therefore

$$\mathrm{Spec}(A(t_k)) = a(t_k)\,\mathrm{Spec}(\varphi_{t_k} A_0 \varphi_{t_k}^{-1}) = a(t_k)\,\mathrm{Spec}(A_0), \quad \forall t_k \in (T_-, \infty).$$

Then, as $A(t_k) \to A_\infty$, we have that

$$\mathrm{Spec}(A_\infty) = a_\infty \mathrm{Spec}(A_0),$$

where $a_\infty = \lim_{k\to\infty} a(t_k)$ (recall that from Lemma 3.8, $a(t)$ is a positive, non-increasing function). $\qquad\square$

Proposition 3.10. $\mathrm{tr}(S(A(t))^2)$ *is strictly decreasing if A_0 is not skew-symmetric. Moreover, $\mathrm{tr}(S(A(t))^2) \to 0$, as $t \to \infty$.*

Proof. Recall that $S(A) = \frac{1}{2}(A + A^t)$, and so

(14) $$\mathrm{tr}(S(A)^2) = \tfrac{1}{2}\|A\|^2 + \tfrac{1}{2}\mathrm{tr}(A^2).$$

Then, as in Proposition 3.4 we have already studied $\frac{d}{dt}\|A\|^2$, we will only analyze $\frac{d}{dt}\mathrm{tr}(A^2)$. By using (7), we obtain

(15) $$\tfrac{d}{dt}\mathrm{tr}(A^2) = \tfrac{d}{dt}\langle A, A^t \rangle = \langle \tfrac{d}{dt}A, A^t \rangle + \langle A, \tfrac{d}{dt}A^t \rangle = 2\langle \tfrac{d}{dt}A, A^t \rangle = -\mathrm{tr}(S(A)^2)\mathrm{tr}(A^2).$$

Therefore, it follows from (9) and (15) that

$$\tfrac{d}{dt}\mathrm{tr}(S(A)^2) = -2\,\mathrm{tr}(S(A)^2)^2 - \tfrac{1}{2}\|[A, A^t]\|^2 \le 0,$$

and if there exists t_0 such that $\frac{d}{dt}|_{t=t_0}\mathrm{tr}(S(A)^2) = 0$, then $A(t_0)$ is a skew-symmetric and so $A(t) = A(t_0)$, for all t. Conversely if A_0 is skew-symmetric, we have that $\frac{d}{dt}\mathrm{tr}(S(A)^2) = 0$. So, $\mathrm{tr}(S(A)^2)$ is strictly decreasing if A_0 is not skew-symmetric.

In addition,

$$\tfrac{d}{dt}\mathrm{tr}(S(A)^2) \le -2\,\mathrm{tr}(S(A)^2)^2,$$

And then $\mathrm{tr}(S(A)^2)$ is dominated by

$$x(t) = \tfrac{1}{2t + (\mathrm{tr}(S(A(0))^2))^{-1}},$$

which is a solution of $\frac{d}{dt}x = -2x^2$. Therefore $\mathrm{tr}(S(A(t))^2) \to 0$, as $t \to \infty$. $\qquad\square$

Recall that if G_{μ_A} is the simply connected solvable Lie group with Lie algebra $(\mathbb{R}^{n+1}, \mu_A)$, then g_{μ_A} denotes the left-invariant Riemannian metric on G_{μ_A} such that $g_{\mu_A}(e) = \langle \cdot, \cdot \rangle$, where e is the identity of the group G_{μ_A} and $\langle \cdot, \cdot \rangle$ is the canonical inner product on $\mathbb{R}^{n+1}$.

Corollary 3.11. *If $A(t) \to A_\infty$, as $t \to \infty$, then A_∞ is a skew-symmetric matrix and for any sequence $t_k \to \infty$ there exists a subsequence of $(G_{\mu_{A(t_k)}}, g_{\mu_{A(t_k)}})$ which converges in the pointed topology to a manifold locally isometric to $(G_{\mu_{A_\infty}}, g_{\mu_{A_\infty}})$, which is flat.*

Proof. By Proposition 3.10, we know that $S(A(t)) \to 0$, as $t \to \infty$, therefore A_∞ is skew-symmetric and then $(G_{\mu_{A_\infty}}, g_{\mu_{A_\infty}})$ is flat (see Remark 3.2).

Finally, since $\mu_{A(t)} \to \mu_{A_\infty}$, by [L3, Corollary 6.20], for any sequence $t_k \to \infty$ there exists a subsequence of $(G_{\mu_{A(t_k)}}, g_{\mu_{A(t_k)}})$ which converges in the pointed topology to a manifold locally isometric to $(G_{\mu_{A_\infty}}, g_{\mu_{A_\infty}})$, which is flat, as shown above. $\qquad\square$

In the following proposition, we prove that under an additional hypothesis, the convergence is actually smooth.

Proposition 3.12. *If* $\mathrm{Spec}(A_0) \nsubseteq i\mathbb{R}$ *and* $A(t) \to A_\infty$, *as* $t \to \infty$, *then* $g_{\mu_{A(t)}} \to g_{\mu_{A_\infty}}$ *smoothly on* $\mathbb{R}^{n+1}$.

Proof. For each μ_A, we define $\psi : \mathbb{R} \oplus \mathbb{R}^n \to G_{\mu_A}$ by

(16)
$$\psi(r,x) = \exp_{\mu_A}(re_0) \exp_{\mu_A}(x), \quad r \in \mathbb{R}, \quad x \in \mathbb{R}^n,$$

where $\exp_{\mu_A} : (\mathbb{R} \oplus \mathbb{R}^n, \mu_A) \to G_{\mu_A}$ is the Lie exponential of G_{μ_A}.

Let $\varphi : (\mathbb{R}^{n+1}, \mu_A) \to (\mathfrak{gl}_{n+1}(\mathbb{R}), [\cdot, \cdot])$ be the linear transformation such that $\varphi(e_0) = X_0$ and $\varphi(e_i) = X_i$, $i = 1, \ldots, n$, where $X_0 = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and $X_i = \begin{pmatrix} 0 & \tilde{e}_i^t \\ 0 & 0 \end{pmatrix}$, $\tilde{e}_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^n$. Then φ is an isomorphism of Lie algebras.

Therefore, under the isomorphism φ, we have that

$$\psi(r,x) = \exp(rX_0) \exp(x), \quad r \in \mathbb{R}, \quad x \in \mathbb{R}^n,$$

where exp is the exponential function of matrices.

Then

$$\psi(r,x) = \exp(rX_0) \exp(x) = \exp(rX_0) \exp(x_1 X_1 + \ldots + x_n X_n),$$

but $\exp(rX_0) = \begin{pmatrix} \exp(rA) & 0 \\ 0 & 1 \end{pmatrix}$ and $\exp(x) = \begin{pmatrix} I & x \\ 0 & 1 \end{pmatrix}$, therefore

$$\psi(r,x) = \begin{pmatrix} \exp(rA) & \exp(rA)x \\ 0 & 1 \end{pmatrix}.$$

It is easy to see that if $\mathrm{Spec}(A) \nsubseteq i\mathbb{R}$ or $A = 0$, then ψ is a diffeomorphism. So, as $\mathrm{Spec}(A_0) \nsubseteq i\mathbb{R}$, we have that $\mathrm{Spec}(A(t)) \nsubseteq i\mathbb{R}$ and $\mathrm{Spec}(A_\infty) \nsubseteq i\mathbb{R}$ or $A_\infty = 0$ by Proposition 3.9, and therefore we have that $g_{\mu_{A(t)}} \to g_{\mu_{A_\infty}}$ smoothly on $\mathbb{R}^n$ (see [L3, Remark 6.11]). $\square$

Remark 3.13. In particular, if μ_{A_0} is completely solvable ($\mathrm{Spec}(\mathrm{ad}_\mu x) \subseteq \mathbb{R}$, for all x), then the convergence is smooth. This also follows by using Proposition 3.9 and [L3, Corollary 6.20], since $\mu_{A(t)}$ is completely solvable for all t.

Recall that if the norm of the Riemann tensor decays at least as fast as $\frac{C}{t}$, where C is a constant, then the solution of the Ricci flow is a Type-III solution (see Definition 3.14).

Proposition 3.14. *For every* μ_{A_0} *with* $\mathrm{tr}(A_0{}^2) \geq 0$, *the Ricci flow* $g(t)$ *with* $g(0) = g_{\mu_{A_0}}$ *is a Type-III solution, for some constant* C_{n+1} *that only depends on the dimension* $n + 1$.

Proof. In Proposition 3.4, we proved that $\mu_{A(t)}$ is defined for $t \in [0, \infty)$. We observe that, by using (15), if $\mathrm{tr}(A_0{}^2) \geq 0$ then $\mathrm{tr}(A(t)^2) \geq 0$ for all t. Further, in Proposition 3.10, we prove that $\mathrm{tr}(S(A)^2) \leq \frac{1}{2t + (\mathrm{tr}(S(A(0))^2))^{-1}}$, therefore, by using (14), we have that

$$\| \mathrm{Rm}(\mu_A)\| = \|\mu_A\|^2 \| \mathrm{Rm}(\tfrac{\mu_A}{\|\mu_A\|})\| = 2\|A\|^2 \| \mathrm{Rm}(\tfrac{\mu_A}{\|\mu_A\|})\| \leq \frac{4C}{2t + (\mathrm{tr}(S(A(0))^2))^{-1}} \leq \frac{2C}{t},$$

where C is the maximum of the continuous function $\mu \to \| \mathrm{Rm}(\mu)\|$ restricted to the unit sphere of $\mathfrak{L}_{n+1}$. $\square$

The question that naturally arises is whether the flow converges. The following section is devoted to study such question.

4. Limit points

In this section, we analyze the ω-limit of the bracket flow $\mu_{A(t)}$ (i.e. the set of limit points of sequences under the bracket flow). To do this, we consider two cases: when $\mathrm{tr}(A_0) = 0$ (i.e., μ_{A_0} is unimodular) and when $\mathrm{tr}(A_0) \neq 0$.

Let us first suppose that $\mathrm{tr}(A_0) = 0$.

We consider the functional $F(A) = \|[A, A^t]\|^2$, which is, in fact, the square norm of the moment map of the conjugation action of the real reductive group $\mathrm{GL}_n(\mathbb{R})$ on the vector space $\mathfrak{gl}_n(\mathbb{R})$, and we compute its gradient:

$$
\begin{aligned}
\langle grad(F)_A, B \rangle &= \tfrac{d}{dt}|_{t=0} F(A + tB) = \tfrac{d}{dt}|_{t=0} \|[A + tB, A^t + tB^t]\|^2 \\
&= \tfrac{d}{dt}|_{t=0} \langle [A + tB, A^t + tB^t], [A + tB, A^t + tB^t] \rangle \\
&= 2\langle [A, A^t], \tfrac{d}{dt}|_{t=0}[A + tB, A^t + tB^t] \rangle \\
&= 2\langle [A, A^t], [B, A^t] + [A, B^t] \rangle \\
&= 4\langle [A, A^t], [B, A^t] \rangle = -4\langle [A, [A, A^t]], B \rangle.
\end{aligned}
$$

Thus, $grad(F)_A = -4[A, [A, A^t]]$ and the negative gradient flow of F is given by

(17) $$\frac{d}{dt}\bar{A}(t) = 4[\bar{A}(t), [\bar{A}(t), \bar{A}(t)^t]].$$

Observe that $\|\bar{A}\|$ is a decreasing function. Indeed,

$$\frac{d}{dt}\|\bar{A}\|^2 = 2\langle \bar{A}', \bar{A} \rangle = 8\langle [\bar{A}, [\bar{A}, \bar{A}^t]], \bar{A} \rangle = -8\|[\bar{A}, \bar{A}^t]\|^2,$$

as $\langle [\bar{A}, [\bar{A}, \bar{A}^t]], \bar{A} \rangle = -\|[\bar{A}, \bar{A}^t]\|^2$. So, $\bar{A}(t)$ has a limit point A^1_∞ and then we have that there exists the limit of $\bar{A}(t)$, as $t \to \infty$ and it is unique (see [KMP, Introduction]). In addition, if $\bar{A}(t) \to A^1_\infty$, we have two cases:

- If $A^1_\infty \neq 0$, then $\lim_{t\to\infty} \frac{\bar{A}(t)}{\|\bar{A}(t)\|}$ exists and $\lim_{t\to\infty} \frac{\bar{A}(t)}{\|\bar{A}(t)\|} = \frac{A^1_\infty}{\|A^1_\infty\|}$.
- If $A^1_\infty = 0$, then by [KMP, Theorem 7.1], $\lim_{t\to\infty} \frac{\bar{A}(t)}{\|\bar{A}(t)\|}$ exists.

If A_0 is nilpotent, then μ_{A_0} turns out to be nilpotent and so the bracket flow starting at μ_{A_0} has been studied in [L2] (see Theorem 2.6). Therefore, we assume that A_0 is not nilpotent.

Lemma 4.1. *Assume that $\mathrm{tr}(A_0) = 0$ and A_0 is not nilpotent. Let $\mu_{A(t)}$ be the bracket flow starting at μ_{A_0} and let $\bar{A}(t)$ be the negative gradient flow (17) starting at A_0. Then the limit of $\frac{A(t)}{\|A(t)\|}$ exists and*

$$\lim_{t\to\infty} \frac{A(t)}{\|A(t)\|} = \lim_{t\to\infty} \frac{\bar{A}(t)}{\|\bar{A}(t)\|}.$$

Proof. We prove that, up to scaling and reparameterization of the time, the bracket flow $A(t)$ starting at A_0 is $\bar{A}(t)$, the solution of (17) starting at A_0, i.e. we want to show that there exist $c(t)$ and $\tau(t)$ such that $A(t) = c(t)\bar{A}(\tau(t))$.

Let $c(t)$ and $\tau(t)$ be solutions of the following system of differential equations with initial conditions:

$$
\begin{aligned}
c'(t) &= -\mathrm{tr}(S(\bar{A}(\tau(t))^2))c(t)^3, & c(0) &= 1, \\
\tau'(t) &= \tfrac{1}{8}c(t)^2, & \tau(0) &= 0.
\end{aligned}
$$

It is easy to see that $c(t)$ and $\tau(t)$ are defined for all t, and with a simple calculation it is easy to verify that $c(t)\bar{A}(\tau(t))$ is a solution of the equation (7), therefore by uniqueness

$$A(t) = c(t)\bar{A}(\tau(t)), \quad \forall t \in [0, \infty).$$

If $\tau(t) \to \infty$ then

$$\lim_{t\to\infty} \frac{A(t)}{\|A(t)\|} = \lim_{t\to\infty} \frac{\bar{A}(\tau(t))}{\|\bar{A}(\tau(t))\|} = \lim_{t\to\infty} \frac{\bar{A}(t)}{\|\bar{A}(t)\|}.$$

We suppose that $\tau(t) \to L$, $L < \infty$, as $t \to \infty$, then

$$\lim_{t\to\infty} \frac{A(t)}{\|A(t)\|} = \lim_{t\to\infty} \frac{\bar{A}(\tau(t))}{\|\bar{A}(\tau(t))\|} = \frac{\bar{A}(L)}{\|\bar{A}(L)\|}.$$

This implies that $\frac{\bar{A}(L)}{\|\bar{A}(L)\|}$ is an algebraic soliton, since it is the limit of a normalized bracket flow (see [LL, Proposition 4.1]). As, A_0 is not nilpotent and $\bar{A}(t)$ is conjugated to A_0, for each t, we have then $\frac{\bar{A}(L)}{\|\bar{A}(L)\|}$ is normal (see Proposition 3.3), i.e. $\bar{A}(L)$ is normal. So, $\bar{A}(t) = \bar{A}(L)$, for all $t \geq L$, by (17).

Therefore,

$$\lim_{t\to\infty} \frac{A(t)}{\|A(t)\|} = \lim_{t\to\infty} \frac{\bar{A}(\tau(t))}{\|\bar{A}(\tau(t))\|} = \lim_{t\to\infty} \frac{\bar{A}(t)}{\|\bar{A}(t)\|},$$

as was to be shown. $\qquad\qquad\square$

Remark 4.2. It follows from Lemma 4.1 and [L2, Section 7] that if μ_{A_0} is unimodular, i.e. $\operatorname{tr}(A_0) = 0$, then the ω-limit of $\frac{A(t)}{\|A(t)\|}$ is a single point.

Lemma 4.3. *If* $\operatorname{tr}(A_0) \neq 0$, *then* $A(t) \to 0$, *as* $t \to \infty$.

Proof. We know that $A(t) = a(t)\varphi_t A_0 \varphi_t^{-1}$ by Lemma 3.8, therefore

$$\operatorname{tr}(A(t)) = a(t)\operatorname{tr}(A_0).$$

If $A(t_k) \to A_\infty$, then

$$a(t_k)\operatorname{tr}(A_0) = \operatorname{tr}(A(t_k)) \to \operatorname{tr}(A_\infty) = 0,$$

so, as $\operatorname{tr}(A_0) \neq 0$, we have that $a(t_k) \to 0$.

On the other hand,

$$\operatorname{Spec}(A(t_k)) = a(t_k)\operatorname{Spec}(A_0) \to \operatorname{Spec}(A_\infty),$$

and so $\operatorname{Spec}(A_\infty) = 0$. Then $A_\infty = 0$, since A_∞ is a skew-symmetric matrix. $\qquad\square$

By using the two previous lemmas, we can prove the following theorem, which provides information about the ω-limit of μ_{A_0}, for any $A_0 \in \mathfrak{gl}_n(\mathbb{R})$.

Theorem 4.4. *The ω-limit of μ_{A_0} is a single point, for any $A_0 \in \mathfrak{gl}_n(\mathbb{R})$.*

Proof. By Lemma 4.3, we have that if $\operatorname{tr}(A_0) \neq 0$, then $A(t) \to 0$, as $t \to \infty$. If $\operatorname{tr}(A_0) = 0$, we know by Remark 4.2 that

$$\lim_{t\to\infty} \frac{A(t)}{\|A(t)\|} = A_\infty^2.$$

Then, we have that $A(t) \to A_\infty$, as $t \to \infty$. Indeed, the norm of $A(t)$ decreases and therefore $\lim_{t\to\infty} \|A(t)\| = \alpha$. If $\alpha = 0$, then $A(t) \to 0$ and if $\alpha > 0$, we have that

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty} \alpha \frac{A(t)}{\|A(t)\|} = \alpha A_\infty^2,$$

which completes the proof. $\qquad\qquad\square$

All results obtained so far can be summarized in the following theorem.

Theorem 4.5. *Given $A_0 \in \mathfrak{gl}_n(\mathbb{R})$, consider the bracket flow $\mu_{A(t)}$ starting at μ_{A_0} and $g(t)$ the Ricci flow starting at $g_{\mu_{A_0}}$. Then,*
- *(i) $g(t)$ is defined for $t \in (T_-, \infty)$, where $-\infty < T_- < 0$.*
- *(ii) The ω-limit of μ_{A_0} is a single point.*
- *(iii) For any sequence $t_k \to \infty$, there exists a subsequence of $(G_{\mu_{A(t_k)}}, g_{\mu_{A(t_k)}})$ which converges in the pointed topology to a manifold locally isometric to $(G_{\mu_{A_\infty}}, g_{\mu_{A_\infty}})$, which is flat.*

(iv) *If* $\mathrm{Spec}(A_0) \not\subseteq i\mathbb{R}$, *then* $g_{\mu_{A(t)}} \to g_{\mu_{A_\infty}}$ *smoothly on* $\mathbb{R}^n$.

(v) *If* $\mathrm{tr}(A_0{}^2) \geq 0$, *the Ricci flow* $g(t)$ *with* $g(0) = g_{\mu_{A_0}}$ *is a Type-III solution, for some constant* C_{n+1} *that only depends on the dimension* $n + 1$.

Example 4.6. Let $A_0 = \begin{pmatrix} 0 & x_0 \\ y_0 & 0 \end{pmatrix}$. It is easy to see that the family of matrices of this kind is invariant under the flow (7), which is equivalent to the following ODE system for the variables $x(t), y(t)$:

$$(18) \qquad \begin{cases} x' = x(x+y)(-\frac{3}{2}x + \frac{1}{2}y), & x(0) = x_0, \\ y' = y(x+y)(-\frac{3}{2}y + \frac{1}{2}x), & y(0) = y_0. \end{cases}$$

The phase plane for this system is displayed in Figure 1, as computed in Maple. It is easy to
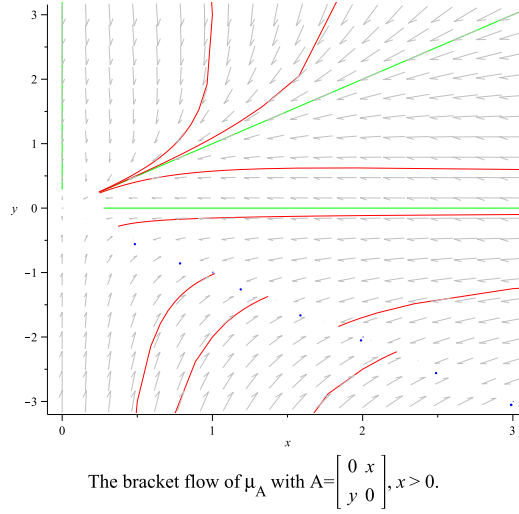

The bracket flow of μ_A with $A = \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}, x > 0$.

FIGURE 1.

see that it is enough to assume $0 \leq x_0$, since if (x, y) is the solution starting at (x_0, y_0), then $(-x, -y)$ is the solution starting at $(-x_0, -y_0)$.

Regarding the interval of definition, the solutions remain in a compact subset and so they are defined in $[0, \infty)$.

The solutions converge to the points $(x_\infty, -x_\infty)$, which are precisely the fixed points of the system and correspond to skew-symmetric matrices (which in turn correspond to flat metrics). Also, we observe that points of the form (x_0, x_0), $(x_0, 0)$ and $(0, y_0)$ correspond to algebraic solitons (they are symmetric or special nilpotent matrices). Despite the fact that the solutions in the upper half-plane converge to 0, we can see from the figure that they are approaching the soliton line $y = x$, so considering a suitable normalization we may be able to obtain convergence of those solutions to a non-flat algebraic soliton. This will be the topic of the next section.

5. NORMALIZING BY THE BRACKET NORM

According to Theorem 4.5 (iii), for any sequence $t_k \to \infty$ there exists a subsequence in which the Ricci flow converge in the pointed topology to a flat manifold. In order to avoid this type of convergence and get a more interesting limit, we consider different normalizations of the flow. In this section, we study the normalized bracket flow by the bracket norm, i.e. if $\mu_{A(t)}$ is the bracket flow starting at μ_{A_0}, we will study $\frac{A(t)}{\|A(t)\|}$. We use the positive, non-increasing function obtained in Section 3 to determine which limits correspond to flat manifolds. Before stating

the theorem of convergence, we demonstrate the following technical lemma. From now on, let $B(t) = \frac{A(t)}{\|A(t)\|}$.

Lemma 5.1. *The following evolution equations along the normalized flow by the bracket norm hold:*

(i) $\frac{d}{dt}\operatorname{tr}(B) = \frac{1}{2}\|A\|^2\|[B,B^t]\|^2\operatorname{tr}(B)$,

(ii) $\frac{d}{dt}\operatorname{tr}(B^2) = \|A\|^2\|[B,B^t]\|^2\operatorname{tr}(B^2)$.

Proof. To prove (i), we use (7) and (9). Part (ii) follows from (15) and (9). $\qquad\square$

Theorem 5.2. *For any sequence $t_k \to \infty$, there exists a subsequence of $(G_{\mu_{B(t_k)}}, g_{\mu_{B(t_k)}})$ converging in the pointed topology to an algebraic soliton $(G_{\mu_{B_\infty}}, g_{\mu_{B_\infty}})$. Moreover, the following conditions are equivalent:*

(i) $\operatorname{Spec}(A_0) \subseteq i\mathbb{R}$.

(ii) $(G_{\mu_{B_\infty}}, g_{\mu_{B_\infty}})$ *is flat.*

Proof. As $\|B(t)\| = 1$, every sequence has a convergent subsequence, i.e., $B(t_k)$ converge to B_∞, which is an algebraic soliton (see Corollary 3.7). By using (13), we have that

$$(19) \qquad\qquad \operatorname{Spec}(B(t_k)) = \operatorname{Spec}(\tfrac{A(t_k)}{\|A(t_k)\|}) = \tfrac{a(t_k)}{\|A(t_k)\|}\operatorname{Spec}(A_0).$$

If $\operatorname{Spec}(A_0) \subseteq i\mathbb{R}$, then $\operatorname{tr}(B(0)^2) = \operatorname{tr}\left(\frac{A_0{}^2}{\|A_0\|^2}\right) < 0$, and so by Proposition 5.1 (ii), we have that $\operatorname{tr}(B(t)^2) < 0$ for all t, and $\operatorname{tr}(B(t)^2)$ is a decreasing function. It follows that $\operatorname{tr}(B_\infty{}^2) < 0$ and then B_∞ is normal, as B_∞ is an algebraic soliton (see Proposition 3.3). So, by (19), we have that $\operatorname{Spec}(B_\infty) \subseteq i\mathbb{R}$ and so B_∞ is a skew-symmetric matrix. Conversely, if B_∞ is skew-symmetric, then $\operatorname{Spec}(B_\infty) \subseteq i\mathbb{R}$, so, by using (19), we have that $\operatorname{Spec}(A_0) \subseteq i\mathbb{R}$. $\qquad\square$

Here again, we wonder ourselves what happens with the ω-limit of $\frac{A(t)}{\|A(t)\|}$. Recall that in Section 4 we saw that if $\operatorname{tr}(A_0) = 0$, then the ω-limit of $\frac{A(t)}{\|A(t)\|}$ is a single point. In the following proposition we analyze the case $\operatorname{tr}(A_0) \neq 0$.

Proposition 5.3. *If $\operatorname{tr}(A_0) \neq 0$ and $B(t_k) \to B_\infty$, for some sequence $t_k \to \infty$, then the ω-limit of $\frac{A(t)}{\|A(t)\|}$ is contained in $O(n).B_\infty$.*

Proof. Let A_0 be such that $\operatorname{tr}(A_0) \neq 0$ and we suppose that $B(t_k) \to B_\infty^1$ and $B(s_l) \to B_\infty^2$. We want to see that B_∞^1 and B_∞^2 are conjugate by an orthogonal matrix.

- If $\operatorname{tr}(A_0) < 0$, then $\operatorname{tr}(B(0)) < 0$ and by Proposition 5.1 (i), $\operatorname{tr}(B(t)) < 0$, and therefore $\operatorname{tr}(B(t))$ is a decreasing function and it follows that $\operatorname{tr}(B(t)) < \operatorname{tr}(B(0))$, for all t.
- If $\operatorname{tr}(A_0) > 0$, then $\operatorname{tr}(B(0)) > 0$ and by Proposition 5.1 (i), $\operatorname{tr}(B(t)) > 0$, and therefore $\operatorname{tr}(B(t))$ is an increasing function and it follows that $\operatorname{tr}(B(0)) < \operatorname{tr}(B(t))$, for all t.

Then, $\operatorname{tr}(B_\infty^1) \neq 0$ and $\operatorname{tr}(B_\infty^2) \neq 0$. Furthermore, the function $\operatorname{tr}(B(t))$ is either increasing or decreasing. So, $\operatorname{tr}(B_\infty^1) = \operatorname{tr}(B_\infty^2)$. From this and (13) it follows that

$$\lim_{k\to\infty}\tfrac{a(t_k)}{\|A(t_k)\|}\operatorname{tr}(A_0) = \operatorname{tr}(B_\infty^1) = \operatorname{tr}(B_\infty^2) = \lim_{l\to\infty}\tfrac{a(s_l)}{\|A(s_l)\|}\operatorname{tr}(A_0).$$

and

$$\operatorname{Spec}(B_\infty^1) = \lim_{k\to\infty}\tfrac{a(t_k)}{\|A(t_k)\|}\operatorname{Spec}(A_0) = \lim_{l\to\infty}\tfrac{a(s_l)}{\|A(s_l)\|}\operatorname{Spec}(A_0) = \operatorname{Spec}(B_\infty^2).$$

Finally, we observe that B_∞^1 and B_∞^2 are normal matrices, since $\mu_{B_\infty^1}$ and $\mu_{B_\infty^2}$ are algebraic solitons (see Corollary 3.7), and so, B_∞^1 and B_∞^2 are normal or nilpotent (see Proposition 3.3). As $\operatorname{tr}(B_\infty^1) \neq 0$ and $\operatorname{tr}(B_\infty^2) \neq 0$, they are not nilpotent matrices. Then, we have two normal matrices with the same spectrum, from which it follows that they are conjugate by an orthogonal matrix (see [HK]). $\qquad\square$

6. Negative Curvature

In this section, we are interested in how the curvature evolves along the Ricci flow. We define the sectional curvature K of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$, a Lie algebra endowed with an inner product, as the sectional curvature of (G, g), where G is the simply connected Lie group with Lie algebra $\mathfrak{g}$ and g is the left-invariant metric in G such that $g(0) = \langle \cdot, \cdot \rangle$. In the case of $(\mathbb{R}^{n+1}, \mu_A, \langle \cdot, \cdot \rangle)$, we simply denote it by K_A. We say that a Riemannian manifold has negative curvature, and denote it by $K < 0$, if all sectional curvatures are strictly negative.

Next, we enunciate two results proved by Heintze in [Hn]. Theorems 6.1 and 6.3 give necessary and sufficient conditions for certain solvable Lie algebras with an inner product to have negative curvature and for a solvable Lie algebra to admit an inner product with negative curvature, respectively.

Theorem 6.1. [Hn, Theorem 1] *Let $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ be a solvable Lie algebra with an inner product such that the derived algebra is abelian (i.e., $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ abelian). Then $K < 0$ if and only if the following conditions hold:*

(A) $\dim \mathfrak{g}' = \dim \mathfrak{g} - 1$.
(B) *There exists a unit vector $A_0 \in \mathfrak{g}$, orthogonal to $\mathfrak{g}'$, such that $D_0 : \mathfrak{g}' \to \mathfrak{g}'$ is positive definite, where D_0 is the symmetric part of $\operatorname{ad}_{A_0}|_{\mathfrak{g}'} : \mathfrak{g}' \to \mathfrak{g}'$.*
(C) *If S_0 is the skew-symmetric part of $\operatorname{ad}_{A_0}|_{\mathfrak{g}'}$, then $D_0^2 + [D_0, S_0]|_{\mathfrak{g}'}$ is also positive definite.*

Remark 6.2. We observe that in the case of μ_A, the assumption that the derived algebra is abelian is always true. Furthermore, $K_A < 0$ if and only if conditions (A) - (C) hold. If in addition A is normal and invertible, then $K_A < 0$ if and only if (B) holds, since condition (A) is satisfied as A is invertible and condition (C) follows from (B).

Theorem 6.3. [Hn, Theorem 3] *Let $\mathfrak{g}$ be a solvable Lie algebra. Then the following conditions are equivalent:*

(i) *$\mathfrak{g}$ admits an inner product with negative curvature.*
(ii) *$\dim \mathfrak{g}' = \dim \mathfrak{g} - 1$ and there exists $A_0 \in \mathfrak{g}$ such that $\operatorname{Re}(\operatorname{Spec}(\operatorname{ad}_{A_0}|_{\mathfrak{g}'})) > 0$.*

Remark 6.4. Note that if A is invertible, then G_{μ_A} admits a left-invariant metric with $K < 0$ if and only if either $\operatorname{Re}(\operatorname{Spec}(A)) > 0$ or $\operatorname{Re}(\operatorname{Spec}(A)) < 0$.

Theorem 6.5. *Let $G_{\mu_{A_0}}$ be a solvable Lie group that admits a left-invariant metric with negative curvature. If $\mu_{A(t)}$ is the bracket flow starting at μ_{A_0}, then there exists $s_0 \in \mathbb{R}$ such that $K_{A(t)} < 0$, for all $t \geq s_0$.*

Proof. It is sufficient to prove that the theorem holds for $B(t) = \frac{A(t)}{\|A(t)\|}$, i.e. there exists $t_0 \in \mathbb{R}$ such that $K_{B(t)} < 0$, for all $t \geq t_0$. Indeed, for each t, $\mu_{A(t)}$ and $\mu_{B(t)}$ differ only by scaling.

By assumption, $G_{\mu_{A_0}}$ admits a left-invariant metric with negative curvature, then by using Remark 6.4 we have that either $\operatorname{Re}(\operatorname{Spec}(A_0)) > 0$ or $\operatorname{Re}(\operatorname{Spec}(A_0)) < 0$.

Assume that, after passing to a subsequence, $B(t_k)$ converges to B_∞, as $k \to \infty$. Then, arguing as in Proposition 5.3, we have that B_∞ is normal and

$$\operatorname{Spec}(B_\infty) = \alpha \operatorname{Spec}(A_0), \quad \alpha \in \mathbb{R}, \alpha \neq 0.$$

so, either $\operatorname{Re}(\operatorname{Spec}(B_\infty)) > 0$ or $\operatorname{Re}(\operatorname{Spec}(B_\infty)) < 0$. Then $S(B_\infty)$ is either positive or negative definite. It follows by Remark 6.2 that $K_{B_\infty} < 0$. Thus, there exists $L \in \mathbb{N}$ such that $K_{B(t_k)} < 0$, for all $k \geq L$.

Finally, there must exist t_0 such that $K_{B(t)} < 0$, for all $t \geq t_0$, otherwise we would be able to extract a convergent subsequence $B(t_k)$, whose sectional curvatures are not strictly negative, and this contradicts the previous paragraph. $\square$

We now show that the above theorem is not longer valid in the general solvable case.

Example 6.6. We consider $(\mu_{\lambda,\alpha}, \langle \cdot, \cdot \rangle)$ defined as follows:

$$\mu_{\lambda,\alpha}(e_0, e_i) = \alpha \begin{pmatrix} \lambda & & \\ & 1-\lambda & \\ & & 1 \end{pmatrix} e_i, \quad \mu_{\lambda,\alpha}(e_1, e_2) = e_3,$$

and $\langle \cdot, \cdot \rangle$ the inner product for which $\{e_0, e_1, e_2, e_3\}$ is an orthonormal basis. By [L1, Theorem 4.8], we know that $(\mu_{\lambda,\alpha}, \langle \cdot, \cdot \rangle)$ is an algebraic soliton if and only if $\alpha = \frac{\sqrt{3}}{\sqrt{2(\lambda^2+(1-\lambda)^2+1)}}$. We consider the 2-dimensional plane $\pi = \langle e_1, e_3 \rangle$ and we compute its sectional curvature:

$$K(e_1, e_3) = \|U(e_1, e_3)\|^2 - \langle U(e_1, e_1), U(e_3, e_3) \rangle = \tfrac{1}{4} - \tfrac{3\lambda}{\lambda^2+(1-\lambda)^2+1}.$$

So,

$$K(e_1, e_3) \geq 0 \quad \Leftrightarrow \quad \tfrac{1}{4} - \tfrac{3\lambda}{\lambda^2+(1-\lambda)^2+1} \geq 0 \quad \Leftrightarrow \quad \lambda \leq 2 - \sqrt{3} \quad \text{ó} \quad \lambda \geq 2 + \sqrt{3}.$$

We observe that if $0 < \lambda \leq 2 - \sqrt{3}$, then $0 < 1 - \lambda$, and so $\mathrm{ad}(e_0)$ is a matrix such that $\mathrm{Re}(\mathrm{Spec}(\mathrm{ad}(e_0))) > 0$. Then, Theorem 6.3 said that if $0 < \lambda \leq 2 - \sqrt{3}$, then $(\mu_{\lambda,\alpha}, \langle \cdot, \cdot \rangle)$, with $\alpha = \frac{\sqrt{3}}{\sqrt{2(\lambda^2+(1-\lambda)^2+1)}}$, admits an inner product with negative curvature. On the other hand, since $(\mu_{\lambda,\alpha}, \langle \cdot, \cdot \rangle)$ is an algebraic soliton, if $\mu(t)$ is the bracket flow starting at $\mu_{\lambda,\alpha}$, then $(G_{\mu(t)}, g_{\mu(t)})$ has planes with curvature bigger than or equal to zero.

The next question is what happens with the Ricci flow when we start with a metric whose sectional curvatures are all negative. First, we will introduce a theorem proved by Heintze in [Hn].

Let $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ be a solvable Lie algebra with an inner product such that (A) - (C) of the Theorem 6.1 hold. Then, we have a orthogonal decomposition $\mathfrak{g} = A_0 + [\mathfrak{g}, \mathfrak{g}]$. For $\alpha > 0$, let $(\mathfrak{g}_\alpha, \langle \cdot, \cdot \rangle)$ be the Lie algebra with the same inner product that $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ but with the following modification in the Lie bracket

$$[A_0, X]_\alpha := \alpha[A_0, X], \text{ para todo } X \in \mathfrak{g}' = \mathfrak{g}'_\alpha.$$

Theorem 6.7. [Hn, Theorem 2] *Let $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ be a solvable Lie algebra with an inner product and assume that (A)-(C) hold. Then there exists $\alpha_0 > 0$ such that $(\mathfrak{g}_\alpha, \langle \cdot, \cdot \rangle)$ has negative curvature for all $\alpha \geq \alpha_0$.*

We return to Example 6.6. Let λ be fixed and we consider the bracket flow $\mu(t)$ starting at $\mu_{\alpha,\lambda}$. Then $\mu(t)$ is given by

$$\mu(t)(e_0, e_i) = \alpha(t) \begin{pmatrix} \lambda & & \\ & 1-\lambda & \\ & & 1 \end{pmatrix} e_i, \quad \mu(t)(e_1, e_2) = h(t)e_3,$$

with $\alpha = \alpha(t)$ and $h = h(t)$ that satisfy the following differential equations:

$$\begin{cases} \alpha' = -c_\lambda \alpha^3, & \alpha(0) = \alpha, \\ h' = -\tfrac{3}{2}h^3, & h(0) = 1, \end{cases}$$

where $c_\lambda = (\lambda^2 + (1-\lambda)^2 + 1)$. Furthermore, solving the equations we obtain that $\alpha(t) = \frac{1}{\sqrt{2c_\lambda t + \alpha^{-2}}}$ and $h(t) = \frac{1}{\sqrt{3t+1}}$. Clearly, in this case, the bracket flow converge to a flat metric, but for fixed t, we have that

$$K(e_1, e_3) = \tfrac{h^2}{4} - \lambda\alpha^2 = \tfrac{1}{4(3t+1)} - \tfrac{\lambda}{2c_\lambda t + \alpha_0^{-2}}.$$

Then,

$$K(e_1, e_3) \geq 0 \quad \Leftrightarrow \quad \tfrac{1}{4(3t+1)} \geq \tfrac{\lambda}{2c_\lambda t + \alpha_0^{-2}} \quad \Leftrightarrow \quad (2c_\lambda - 12\lambda)t \geq 4\lambda - \alpha_0^{-2}.$$

Further, $2c_\lambda - 12\lambda = 4((\lambda - 2)^2 - 3)$. So, if $0 < \lambda \leq 2 - \sqrt{3}$, there exists t_0 such that $K(e_1, e_3) \geq 0, \forall t \geq t_0$.

Let λ be such that $0 < \lambda \leq 2 - \sqrt{3}$, and we consider $\mu_{\alpha,\lambda}, \alpha \in \mathbb{R}_{>0}$. Then $(\mu_{\alpha,\lambda}, \langle \cdot, \cdot \rangle)$ is a solvable Lie algebra with an inner product that satisfies (A) - (C). By Theorem 6.7, we know that there exists $\alpha_0 > 0$ such that $((\mu_{\alpha,\lambda})_{\alpha_0}, \langle \cdot, \cdot \rangle)$ has negative curvature. Then, $(\mu_{\alpha\alpha_0,\lambda}, \langle \cdot, \cdot \rangle)$ has a negative curvature. On the other hand, we know that if $\mu(t)$ is the bracket flow starting at $\mu_{\alpha\alpha_0,\lambda}$ there exists t_0 such that $\forall t \geq t_0$, $(G_{\mu(t)}, g_{\mu(t)})$ has planes with curvature bigger than or equal to zero.

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